Name ....

Further Maths

Differentiation

Summer Workbook

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Edexcel AS Mathematics: Differentiation

Section 1: Introduction to differentiation

Notes and Examples

These notes contain the following subsections:

What is differentiation?

In this section, you will be studying the relationship between the position of a point on a curve and the gradient of the curve.

Straight lines are, by definition, lines of constant gradient. Curves, on the other hand, have varying gradient – the gradient depends on whereabouts you are on the curve. Differentiation is the process of finding the gradient at any point on a curve from the equation of the curve.

Differentiation, together with its reverse process, called integration, form the branch of mathematics called calculus. The discovery of calculus (Made in the 17th century by Isaac Newton in England and, independently, by Gottfried von Leibnitz in Germany) was one of the most significant advances in the history of mathematics and science, and was crucial to unlocking the mathematical basis of our planetary system.

Differentiation is the process of finding the gradient function, or derivative, or derived function. Given an equation for y in terms of x, the gradient function or derivative is written $\frac{\mathrm{d}y}{\mathrm{d}x}$, and gives the gradient of the curve in terms of x.

Investigating gradients

You can investigate how the gradients of chords approach the gradient of a tangent using graphing software, or the Explore resource **The gradient of a curve** available on Integral. You can then go on to investigate the pattern in the value of the gradient at different points on a curve.

Rules for finding derivatives

If y is a polynomial function (made up of powers of x), you may have been able to spot patterns in the gradient function by investigating tangents.

This leads to the following rules which will enable you to find the derivative $\frac{dy}{dx}$:

The derivative of x^n is nx^{n-1},

The derivative of kx^n is knx^{n-1},

The derivative of a constant is zero.

The derivative of a sum (or difference) is the sum (or difference) of the derivatives.

You will learn how these results are formally proved in a later section.

Example 1

Differentiate $y = 2x^3 - 5x^2 + 4$

Solution

The derivative of $2x^3$ is $2 \times 3x^2 = 6x^2$

The derivative of $5x^2$ is $5 \times 2x = 10x$

The derivative of 4 is 0.

So $\frac{dy}{dx} = 6x^2 - 10x$.

The next example involves an expression which is the product of two functions. You cannot differentiate this by differentiating each function separately and then multiplying the results, i.e. the derivative of a product of two functions is not the product of the derivatives! So with examples involving brackets, you will need to multiply out the brackets first. (There is a rule for differentiating the product of two functions, but you do not need to know this yet.)

Example 2

(a) Find the derivative of $y = (x-2)(x^2+1)$.
(b) Hence find the gradient of the curve at the point (2, 0).
(c) Find the coordinates of the points where the gradient is zero.

Solution

(a) $y=(x-2)(x^2+1)=x^3-2x^2+x-2$

$$\frac{dy}{dx}=3x^2-4x+1$$

(b) Substituting $x=2$ into the gradient function,

$$\frac{dy}{dx}=3\times 2^2-4\times 2+1=5$$

so the gradient of the curve at (2, 0) is 5.

(c) The gradient of the curve is zero when $\frac{dy}{dx}=0$

$$\Rightarrow 3x^2-4x+1=0$$
$$\Rightarrow (3x-1)(x-1)=0$$
$$\Rightarrow x=\tfrac{1}{3} \text{ or } x=1$$

Now calculate the y coordinates for these values of x.

When $x=\frac{1}{3}$, $y=\left(\frac{1}{3}-2\right)\left(\frac{1}{9}+1\right)=-\frac{5}{3}\times\frac{10}{9}=-\frac{50}{27}$

When $x=1$, $y=(1-2)(1^2+1)=-2$.

So the points on the curve with gradient zero are (1, –2) and $\left(\frac{1}{3},-\frac{50}{27}\right)$

The points where the gradient is zero are called the turning points or stationary points of the curve. You will look at such points in more detail in Section 2.

The next example involves the quotient of two functions (i.e. one function divided by another). As with products, the derivative of a quotient is not the quotient of the derivatives. You need to divide the fraction first.

Example 3

Differentiate $\dfrac{3x^2-4x^4}{2x}$.

Solution

$$y=\frac{3x^2-4x^4}{2x}=\frac{3x^2}{2x}-\frac{4x^4}{2x}$$
$$=\tfrac{3}{2}x-2x^3$$

$$\frac{dy}{dx} = \tfrac{3}{2} - 6x^2$$

Rates of change

In the work you have done so far, you have looked at the derivative of a function as representing the gradient of the graph of that function. However, more generally, the derivative $\frac{dy}{dx}$ represents the rate of change of y with respect to x. The variables need not be y and x – they can be any letter, representing any quantity.

Finding a rate of change is no different from finding the gradient of a graph – in both cases you differentiate. However, you must make sure that if letters other than y and x are being used, you use the same letters.

e.g. if you are given that $s = t^2 + 3t - 1$, and you are asked to find the rate of change of s with respect to t, you are finding $\frac{ds}{dt}$, not $\frac{dy}{dx}$.

One important application of rates of change is in the motion of a particle. If you study Mechanics you will learn more about this.

Finding tangents and normals to curves

The gradient of a tangent to a curve at a particular point is the same as the gradient of the curve at that point. So to find the equation of a tangent to a curve, you first need to find the gradient m of the curve via differentiation. You can then substitute m and the coordinates (x_1, y_1) of the point on the curve into the formula:

$$y - y_1 = m(x - x_1)$$

Example 4

Find the equation of the tangent to the curve $y = 2x^3 - 3x$ at the point with x-coordinate 1.

Solution

To find the gradient of the tangent, first differentiate the equation of the curve and substitute the x-coordinate

$$y = 2x^3 - 3x \Rightarrow \frac{dy}{dx} = 6x^2 - 3$$

At the point with x-coordinate 1

$$\frac{dy}{dx}=\left(6\times1^2\right)-3=3$$

The gradient of the tangent is therefore 3.

Find the y-coordinate of the point where $x = 1$ by substituting into the equation of the curve.

$y=2x^3-3x$

When $x=1$, $y=\left(2\times1^3\right)-\left(3\times1\right)=2-3=-1$

Use the formula for the equation of a line with $m=3$, $x_1=1$ and $y_1=-1$

$y-y_1=m(x-x_1)$

$y-(-1)=3(x-1)$

$y+1=3x-3$

$y=3x-4$

This is the required equation of the tangent.

The normal to a curve is the line perpendicular to the tangent. Remember that the gradient of a line perpendicular to a line with gradient m is m', where

$$m'=-\frac{1}{m}.$$

Example 5

Show that the normal to the curve $y=2x^2-3x$ at the point (1, –1) passes through the origin.

Solution

First, find the gradient of the tangent by differentiating y.

$$y=2x^2-3x\Rightarrow\frac{dy}{dx}=4x-3$$

At the point with x–coordinate 1, $\frac{dy}{dx}=\left(4\times1\right)-3=1$

Gradient of the tangent = 1, so gradient of the normal = $-\frac{1}{1}=-1$

Use the formula for the equation of a line with $m = -1$, $x_1 = 1$ and $y_1 = -1$

$$y - y_1 = m(x - x_1)$$
$$y - (-1) = -1(x - 1)$$
$$y + 1 = -x + 1$$
$$y = -x$$

This line passes through the origin.

Example 6

A curve has equation $y = x^3 - x^2 + x + 2$.

(a) Find the x-coordinates of the points on the curve with gradient 6.
(b) Find the x-coordinates of the points on the curve for which the gradient of the normal is $-\frac{1}{2}$.

Solution

$$y = x^3 - x^2 + x + 2 \Rightarrow \frac{dy}{dx} = 3x^2 - 2x + 1$$

(a) When gradient = 6

$$3x^2 - 2x + 1 = 6$$
$$\Rightarrow 3x^2 - 2x - 5 = 0$$
$$\Rightarrow (3x - 5)(x + 1) = 0$$
$$\Rightarrow x = \tfrac{5}{3} \text{ or } x = -1$$

(b) Gradient of normal = $-\frac{1}{2} \Rightarrow$ gradient of curve = 2
When gradient = 2

$$3x^2 - 2x + 1 = 2$$
$$\Rightarrow 3x^2 - 2x - 1 = 0$$
$$\Rightarrow (3x + 1)(x - 1) = 0$$
$$\Rightarrow x = -\tfrac{1}{3} \text{ or } x = 1$$

Edexcel AS Mathematics: Differentiation

Section 1: Introduction to differentiation

Exercise level 1

1. Differentiate with respect to x:

   (a) $f(x) = 2x + 1$

   (b) $f(x) = x^3 - 5x$

   (c) $f(x) = x(x + 2)$.

2. For the curve $y = 2x^3 - 3x^2 + x$

   (a) Find $\frac{dy}{dx}$

   (b) Find the gradient of the curve at the point where $x = -2$.

3. Given that $y = 12x - x^3$,

   (a) Find the gradient of the curve at the origin.

   (b) Find the coordinates of the two points where the gradient is zero.

4. The displacement s metres of a particle from a point O after t seconds is given by the equation $s = t^3 - 3t^2 - 9t$.

   (a) Find the rate of change of s with respect to t.

   (b) Hence find the time at which the rate of change of the displacement is zero.

   (c) Explain what happens to the particle when the rate of change of the displacement is zero.

5. Find the equation of the tangent to the curve $y = x^4 - x + 1$ at the point with x–coordinate 1.

6. (a) Show that the equation of the normal to the curve $y = x^2 - x$ at the point (3, 6) is $x + 5y = 33$.

   (b) Find the coordinates of the point where the normal meets the x–axis.

Worked Examples

Worked Examples

Worked Examples

Edexcel AS Mathematics: Differentiation

Section 1: Introduction to differentiation

Exercise level 1 solutions

1. (a) $f(x) = 2x + 1$
   $f'(x) = 2$

   (b) $f(x) = x^3 - 5x$
   $f'(x) = 3x^2 - 5$

   (c) $f(x) = x(x+2) = x^2 + 2x$
   $f'(x) = 2x + 2$

2. (a) $y = 2x^3 - 3x^2 + x$
   $\frac{dy}{dx} = 6x^2 - 6x + 1$

   (b) When $x = -2$, gradient $= 6(-2)^2 - 6(-2) + 1$
   $= 24 + 12 + 1$
   $= 37$

3. (a) $y = 12x - x^3$
   $\frac{dy}{dx} = 12 - 3x^2$
   When $x = 0$, $\frac{dy}{dx} = 12$
   The gradient of the curve at the origin is 12.

   (b) When gradient is zero, $12 - 3x^2 = 0$
   $4 - x^2 = 0$
   $(2 + x)(2 - x) = 0$
   $x = -2$ or $x = 2$
   When x = -2, $y = 12 \times -2 - (-2)^3 = -24 + 8 = -16$
   When x = 2, $y = 12 \times 2 - 2^3 = 24 - 8 = 16$
   The gradient is zero at (-2, -16) and (2, 16).

4. (a) $s = t^3 - 3t^2 - 9t$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 3t^2 - 6t - 9$$

(b) When $\frac{\mathrm{d}s}{\mathrm{d}t} = 0$, $3t^2 - 6t - 9 = 0$

$$t^2 - 2t - 3 = 0$$
$$(t-3)(t+1) = 0$$
$$t = 3 \text{ or } t = -1$$

Since time must be positive, $t = 3$.

The rate of change of the displacement is zero after 3 seconds.

(c) When the rate of change of the displacement is zero, the particle is stationary.

5. $y = x^4 - x + 1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 1$$

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 4 \times 1^3 - 1 = 4 - 1 = 3$

When $x = 1$, $y = 1^4 - 1 + 1 = 1$

The tangent is the straight line with gradient 3 passing through (1, 1).

Equation of tangent is $y - 1 = 3(x-1)$

$$y - 1 = 3x - 3$$
$$y = 3x - 2$$

6. $y = x^2 - x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 1$$

When $x = 3$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \times 3 - 1 = 5$

Gradient of tangent = 5, so gradient of normal $= -\frac{1}{5}$.

The normal is the straight line with gradient $-\frac{1}{5}$ passing through (3, 6).

Equation of normal is $y-6=-\frac{1}{5}(x-3)$

$$5(y-6)=-(x-3)$$
$$5y-30=-x+3$$
$$5y+x=33$$

Where the normal meets the x-axis, y = 0 so x = 33.
The normal meets the x-axis at (33, 0).

Section 1: Introduction to differentiation

Crucial points

1. **Use notation carefully**

   Make sure that you are familiar both with the notation $\frac{dy}{dx}$ (used when you are given y as a function of x) and the notation $f'(x)$ (used when you are given a function $f(x)$).

2. **Use notation in the same way that it is used in the question**
   Example: Differentiate $v = t^2 + 2t$.

   ✗ ***Wrong*** $\frac{dy}{dx} = 2t + 2$.

   ✓ ***Right*** $\frac{dv}{dt} = 2t + 2$, or $\frac{d}{dt}(t^2 + 2t) = 2t + 2$.

   The expression you are differentiating has variables v and t, not y and x, so you are finding $\frac{dv}{dt}$, not $\frac{dy}{dx}$.

3. **When calculating a gradient or tangent to a curve, make sure you get the coordinates the right way round**
   Example: Find the gradient of the curve $y = x^2 + 2x - 3$ when it crosses the x–axis.

   ✗ ***Wrong*** $\frac{dy}{dx} = 2x + 2$. When $x = 0$, $\frac{dy}{dx} = 2 \times 0 + 2 = 0$.

   ✓ ***Right*** $\frac{dy}{dx} = 2x + 2$. Curve crosses x axis when $y = 0$,
   $\Rightarrow x^2 + 2x - 3 = (x - 1)(x + 3) = 0$
   $\Rightarrow x = 1$, $\frac{dy}{dx} = 2 \times 1 + 2 = 4$,
   or $x = -3$, $\frac{dy}{dx} = 2 \times (-3) + 2 = -4$.

4. **Remember the relationship between the gradients of perpendicular lines**
   When finding the gradient of a normal, you need to first find the gradient of the tangent using differentiation, and then use the relationship $m_1 m_2 = -1$. Always show clearly that you are using this relationship.

5. **Draw a diagram if needed**
   Questions on tangents and normals may go on to ask for other coordinate geometry work such as finding where lines cross. If this is the case, a diagram is very helpful.

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Section 1: Introduction to differentiation

Exercise level 2

1. Given that $y = x^3 + 2x^2$, find $\frac{dy}{dx}$.

   Hence find the x–coordinates of the two points on the curve where the gradient is 4.

2. (a) Show that the point (1, 2) lies on both the curves $y = 2x^3$ and $y = 3x^2 - 1$.
   (b) Show that the curves have the same gradient at this point.
   (c) What do these results this tell you about the two curves?

3. Find $\frac{dy}{dx}$ if:

   (a) $y = (x^2 + 1)(x - 1)$

   (b) $y = (x - 1)(x + 1)(x - 2)$

4. A ball is thrown from a window. The height h metres of the ball above the ground after t seconds is given by $h = 12 + 11t - 5t^2$. Find the rate of change of the height of the ball at the instant when it hits the ground.

5. A curve has equation $y = ax^3 + bx$, where a and b are constants. At the point where $x = 1$, the y–coordinate is 8 and the gradient is 12. Find the values of a and b.

6. (a) Show that the tangent to the curve $y = x^3 + x + 2$ at the point P with x–coordinate 1 passes through the origin.
   (b) Find the equation of the normal at this point.
   (c) Given that the normal cuts the x–axis at the point Q, find the area of triangle OPQ.

7. (a) For the curve $y = ax^2 + bx + c$, find the equation of the tangent when $x = p$.
   (b) Find the equation of the tangent from (a), in the case where $b = 0$.
   (c) Explain by reference to the curve $y = ax^2 + bx + c$ why the tangent at $x = 0$ is unchanged for all values of a if $b = 0$.

8. (a) Show that the graphs

$$y = \tfrac{1}{3}x^3 + 2x + 1 \qquad \text{(A)}$$

$$y = x^2 - \tfrac{1}{2}x + 1 \qquad \text{(B)}$$

cross at the point P with coordinates (0, 1).

(b) Find the gradients of the two curves at P.

(c) What can you deduce about the two curves from your results in (b) above?

(d) Show that for any value of a, the curve $y = ax^2 - \tfrac{1}{2}x + 1$ crosses the curve (A) at a constant angle.

Worked Examples

Worked Examples

Worked Examples

Worked Examples

Worked Examples

Edexcel AS Mathematics: Differentiation

Section 1: Introduction to differentiation

Exercise level 2 solutions

1. $y = x^3 + 2x^2$

   $\frac{dy}{dx} = 3x^2 + 4x$

   When gradient is 4, $3x^2 + 4x = 4$

   $3x^2 + 4x - 4 = 0$

   $(3x-2)(x+2) = 0$

   $x = \frac{2}{3}$ or $x = -2$

2. (a) When x = 1, $y = 2x^3 = 2\times 1^3 = 2$

   When x = 1, $y = 3x^2 - 1 = 3\times 1^2 - 1 = 2$

   so the point (1, 2) lies on both curves.

   (b) $y = 2x^3$

   $\frac{dy}{dx} = 6x$

   When x = 1, gradient $= 6\times 1 = 6$

   $y = 3x^2 - 1$

   $\frac{dy}{dx} = 6x$

   When x = 1, gradient $= 6\times 1 = 6$

   so the curves have the same gradient at this point.

   (c) The two curves touch each other at (1, 2).

3. (a) $y = x^3 - x^2 + x - 1$

   $\Rightarrow \frac{dy}{dx} = 3x^2 - 2x + 1$

   (b) $y = (x^2-1)(x-2)$

   $= x^3 - 2x^2 - x + 2$

   $\Rightarrow \frac{dy}{dx} = 3x^2 - 4x - 1$

4. When the ball hits the ground, $h=0$

$$12+11t-5t^2=0$$
$$5t^2-11t-12=0$$
$$(5t+4)(t-3)=0$$
$$t=-\tfrac{4}{5} \text{ or } t=3$$

Since t must be positive, $t=3$ when the ball hits the ground.

$h=12+11t-5t^2$

Rate of change of height $\frac{dh}{dt}=11-10t$

When $t=3$, rate of change of height $11-30=-19$ metres per second.

5. $y=ax^3+bx$

When x = 1, $y=a+b \quad \Rightarrow a+b=8$

$\frac{dy}{dx}=3ax^2+b$

When x = 1, gradient $=3a+b \quad \Rightarrow 3a+b=12$

$$3a+b=12$$
$$a+b=8$$

Subtracting: $2a=4$

$$a=2, b=6$$

6. (a) $y=x^3+x+2$

$\frac{dy}{dx}=3x^2+1$

When x = 1, $\frac{dy}{dx}=3\times1^2+1=4$

When x = 1, $y=1^3+1+2=4$

The tangent has gradient 4 and passes through the point (1, 4).

Equation of tangent is $y-4=4(x-1)$

$$y-4=4x-4$$
$$y=4x$$

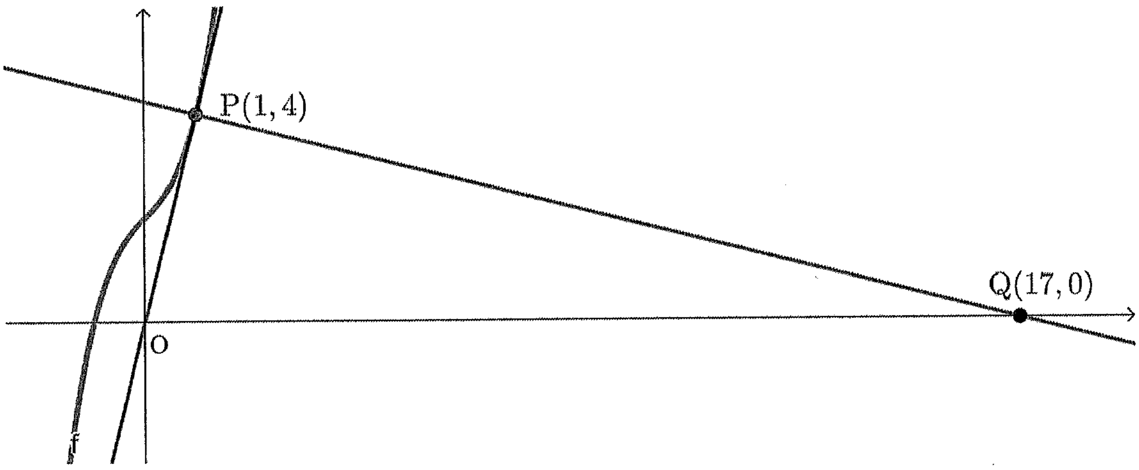
So the tangent passes through the origin.

(b) Gradient of normal $=-\frac{1}{4}$

Equation of normal is $y-4=-\frac{1}{4}(x-1)$

$$4(y-4)=-(x-1)$$
$$4y-16=-x+1$$
$$4y+x=17$$

(c) When y = 0, x = 17, so Q is (17, 0).



Area of triangle $=\frac{1}{2}\times\text{base}\times\text{height}=\frac{1}{2}\times17\times4=34$

7. (a) $x=p\Rightarrow y=ap^2+bp+c$

$\frac{dy}{dx}=2ax+b$

so $x=p\Rightarrow\frac{dy}{dx}=2ap+b$

Equation of tangent is $y-(ap^2+bp+c)=(2ap+b)(x-p)$

$$y=(2ap+b)x-ap^2+c$$

(b) $y=2apx-ap^2+c$

(c) At $x=0$, $p=0$ and so the tangent from (b) is $y=c$

If $b=0$, the equation of the curve is $y=ax^2+c$, so the curve is symmetrical about the y-axis and (0, c) is the vertex of the curve.

So for all values of a, the equation of the tangent at $x=0$ is horizontal and is always $y=c$.

8. (a) At $x=0$, curve (A) gives $y=1$, and curve (B) gives $y=1$, so the curves cross at (0, 1).

(b) For (A), $\frac{dy}{dx}=x^2+2$

so when $x=0$ the gradient of curve = 2.

For (B), $\frac{dy}{dx}=2x-\frac{1}{2}$,

so when $x=0$ the gradient of curve $=-\frac{1}{2}$

(c) The tangents of the two curves are perpendicular, i.e. they cross at right-angles.

(d) For $y=ax^2-\frac{1}{2}x+1$, $\frac{dy}{dx}=2ax-\frac{1}{2}$

so at (0, 1), gradient of curve $=-\frac{1}{2}$ for any value of a. So the tangents are perpendicular for all values of a and so the curves cross at right-angles for all values of a.

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Section 2: Maximum and minimum points

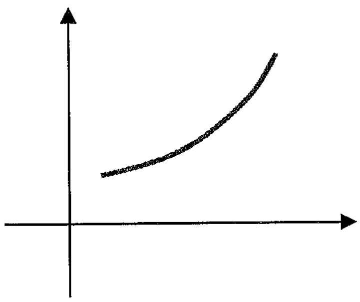
Notes and Examples

These notes contain the following subsections:

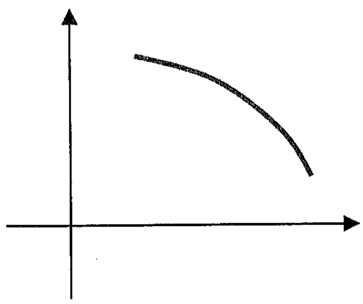
Increasing and decreasing functions

When the gradient $\frac{dy}{dx}$ of a graph is positive, the value of y is increasing.

Similarly, when the gradient is negative, the value of y is decreasing.



$\frac{dy}{dx} > 0$
Increasing function



$\frac{dy}{dx} < 0$
Decreasing function

Example 1

Find the range of values of x for which $y = x^3 - 3x^2 - 9x + 4$ is increasing.

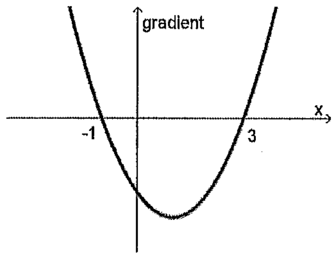
Solution

$y = x^3 - 3x^2 - 9x + 4$

$\frac{dy}{dx} = 3x^2 - 6x - 9$

The function is increasing if $\frac{dy}{dx} > 0$

$$\Rightarrow 3x^2 - 6x - 9 > 0$$
$$\Rightarrow x^2 - 2x - 3 > 0$$
$$\Rightarrow (x-3)(x+1) > 0$$
$$\Rightarrow x < -1 \text{ or } x > 3$$



So the function is increasing for $x < -1$ and for $x > 3$

Turning points

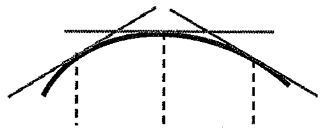
Points on a curve where the tangent is horizontal are called stationary points, or turning points.

At these points, the gradient of the curve is zero, so $\frac{dy}{dx} = 0$.

You will be looking at two types of stationary point:

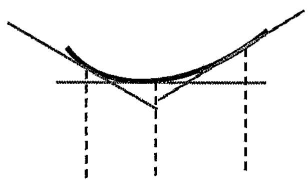
Local maximum

The gradient is positive to the left, zero at the point, and negative to the right



Local minimum

The gradient is negative to the left, zero at the point, and positive to the right.



To distinguish between these, you can test the value of the derivative either side of the stationary point, to see whether the gradient is positive or negative.

This is shown in the next example.

Example 2

Find the stationary points on the curve $y = x^3 - 3x^2 + 1$, investigate their nature, and sketch the curve.

Solution

Step 1: Differentiate the function.

$$y = x^3 - 3x^2 + 1$$
$$\frac{dy}{dx} = 3x^2 - 6x$$

Step 2: Solve $\frac{dy}{dx} = 0$

$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$x = 0 \text{ or } x = 2$$

Step 3: Calculate the y–coordinates for these values of x (called the stationary values).

When $x = 0$, $y = 0^3 - 3 \times 0^2 + 1 = 1$

When $x = 2, y = 2^3 - 3 \times 2^2 + 1 = 8 - 12 + 1 = -3$

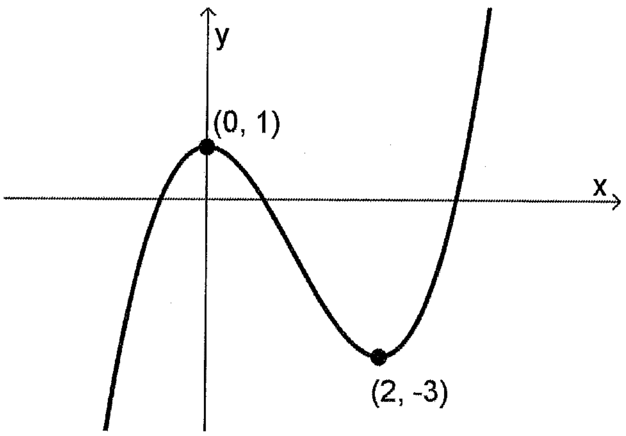
So the stationary points are (0, 1) and (2, –3).

Step 4: Use a table to investigate the sign of $\frac{dy}{dx}$ for values of x either side of the stationary values

x	-1	0	1	2	3
$\frac{dy}{dx}$	9 positive /	0 —	-3 negative \	0 —	9 positive /

So (0, 1) is a local maximum and (2, -3) is a local minimum

Step 5: Sketch the curve.



Sketching the graph of a derivative

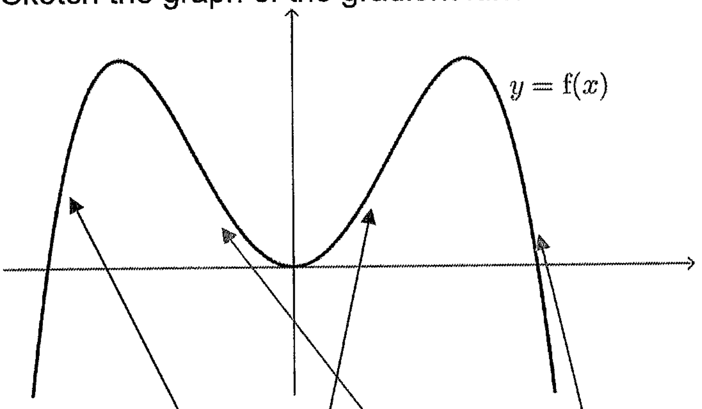
If you have the graph of a function $y = f(x)$, you can sketch the graph of the corresponding gradient function, $y = f'(x)$, by thinking about what is happening at different points on the graph.

- Where there is a turning point, the gradient of $y = f(x)$ is zero so the gradient graph $y = f'(x)$ crosses the x-axis
- Where the graph is increasing, the gradient of $y = f(x)$ is positive so the gradient graph will be above the x-axis
- Where the graph is decreasing, the gradient of $y = f(x)$ is negative so the gradient graph will be below the x-axis

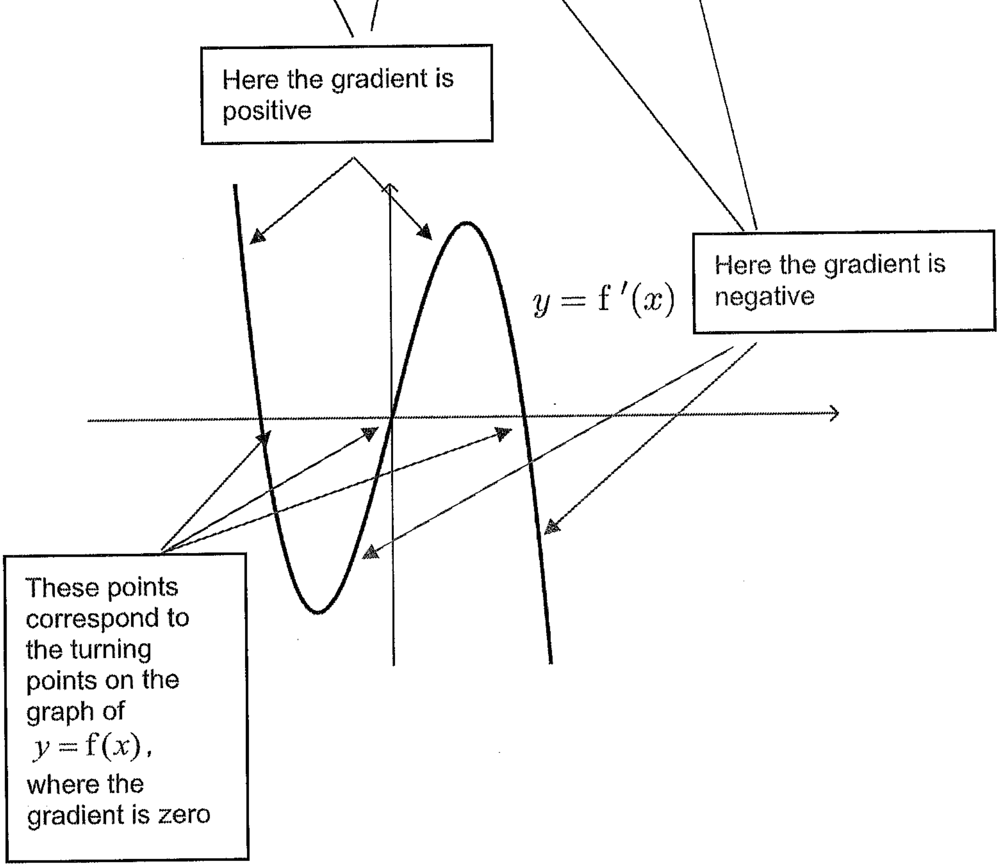
The next example shows how this is done.

Example 3

Sketch the graph of the gradient function of the curve shown below.



Solution



Edexcel AS Mathematics: Differentiation

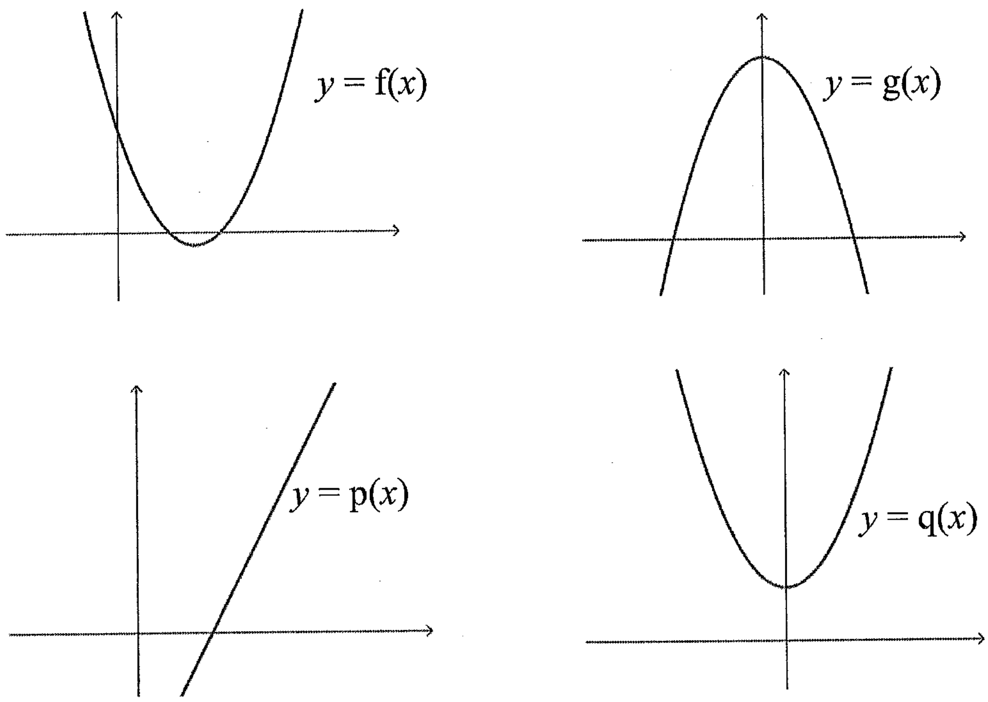
Section 2: Maximum and minimum points

Exercise level 1

1. Find the range of values of x for which $\mathrm{f}(x) = 2x^2 - 3x + 1$ is an increasing function.

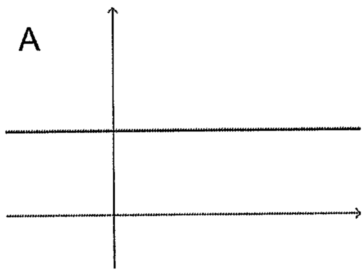
2. Find the range of values of x for which $\mathrm{f}(x) = 4 + 7x - 3x^2$ is a decreasing function.

3. The diagrams below show the graphs of four functions: $\mathrm{f}(x)$, $\mathrm{g}(x)$, $\mathrm{p}(x)$ and $\mathrm{q}(x)$.

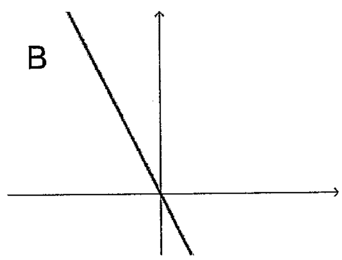


The diagrams below show the gradient functions of $\mathrm{f}(x)$, $\mathrm{g}(x)$, $\mathrm{p}(x)$ and $\mathrm{q}(x)$.
Match the diagrams A, B, C and D to the equations $y = \mathrm{f}'(x)$, $y = \mathrm{g}'(x)$, $y = \mathrm{p}'(x)$ and $y = \mathrm{q}'(x)$.

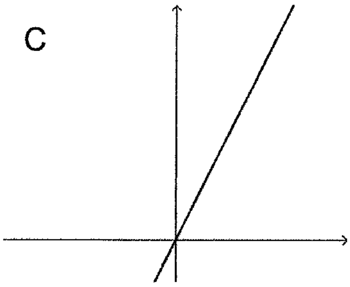
A



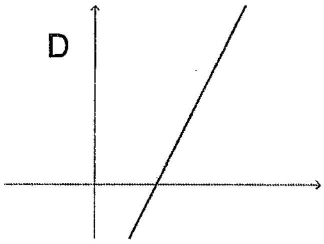
B



C



D



4. A curve has equation $y = x^3 + 6x^2 + 9x$.

(a) Differentiate the function to obtain $\frac{dy}{dx}$.

(b) Find the x-coordinates of the points where $\frac{dy}{dx} = 0$ and hence the coordinates of the turning points on the curve.

(c) By considering the sign of $\frac{dy}{dx}$ on either side of the turning points, determine whether the turning points are maximum or minimum points.

(d) Sketch the curve showing the turning points and points of intersection with the axes.

Worked Examples

Worked Examples

Worked Examples

Edexcel AS Mathematics: Differentiation

Section 2: Maximum and minimum points

Exercise level 1 solutions

1. $f(x) = 2x^2 - 3x + 1$
   $f'(x) = 4x - 3$
   When $f(x)$ is increasing, $f'(x) > 0$
   $\Rightarrow 4x - 3 > 0$
   $\Rightarrow 4x > 3$
   $\Rightarrow x > \frac{3}{4}$

2. $f(x) = 4 + 7x - 3x^2$
   $f'(x) = 7 - 6x$
   When $f(x)$ is decreasing, $f'(x) < 0$
   $\Rightarrow 7 - 6x < 0$
   $\Rightarrow 7 < 6x$
   $\Rightarrow 6x > 7$
   $\Rightarrow x > \frac{7}{6}$

3. The gradient of f(x) starts as negative, becomes zero and then becomes positive. This could be either C or D, but in C the gradient is zero when x = 0, so it must be D.

   The gradient of g(x) starts as positive, is zero when x = 0 and then becomes negative. This is graph B.

   The gradient of p(x) is a constant positive value. This is graph A.

   The gradient of q(x) starts as negative, becomes zero when x = 0, and then becomes positive. This is graph C.

4. (a) $y = x^3 + 6x^2 + 9x$
   $\frac{dy}{dx} = 3x^2 + 12x + 9$

(b) $\frac{dy}{dx} = 0$

$3x^2 + 12x + 9 = 0$

$x^2 + 4x + 3 = 0$

$(x+1)(x+3) = 0$

$x = -1$ or $x = -3$

When x = -1, $y = (-1)^3 + 6(-1)^2 + 9\times -1 = -1 + 6 - 9 = -4$

When x = -3, $y = (-3)^3 + 6(-3)^2 + 9\times -3 = -27 + 54 - 27 = 0$

The turning points are (-1, -4) and (-3, 0)

(c)

x	x < -3	x = -3	-3 < x < -1	x = -1	x > -1
$\frac{dy}{dx}$	+ve /	0 —	-ve \	0 —	+ve /

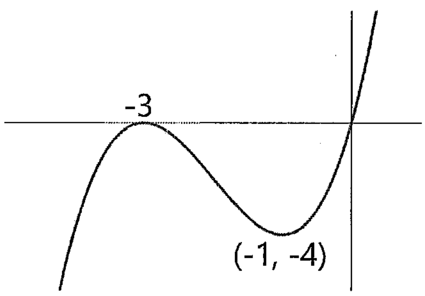
The point (-3, 0) is a maximum point.
The point(-1, -4) is a minimum point.

(d) $y = x^3 + 6x^2 + 9x$

$= x(x^2 + 6x + 9)$

$= x(x+3)^2$

The graph cuts the x-axis at x = 0 and x = -3 (repeated).
The graph cuts the y-axis at y = 0.



Section 2: Maximum and minimum points

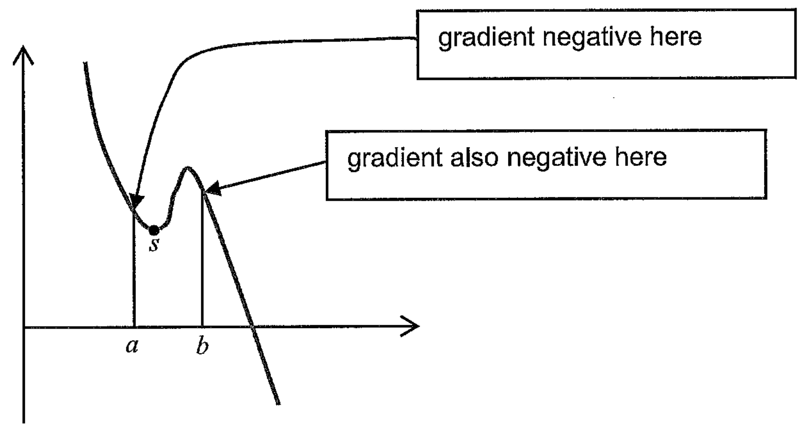
Crucial points

1. **Take care which side of the stationary point you test the gradient**
   When identifying whether a stationary point is a maximum or minimum by testing the sign of gradient either side of the stationary point, make sure you work from left to right, so you find the gradient at a value of x BEFORE the stationary point first, then at a value of x AFTER the stationary point.

   Remember:
   - For a maximum, the gradient is positive before the stationary point and negative after it
   - For a minimum the gradient is negative before the stationary point and positive after it

2. **When testing the gradient either side of a stationary point, make sure the points you test are close enough to the stationary point you are investigating**
   Otherwise, if there are two stationary points very close together, you may come to the wrong conclusion when identifying the stationary point.

   Example



s is a minimum point between a and b, but the gradient is negative at a and negative at b. This error is caused by there being two stationary points close together, both of which are between a and b.

Edexcel AS Mathematics: Differentiation

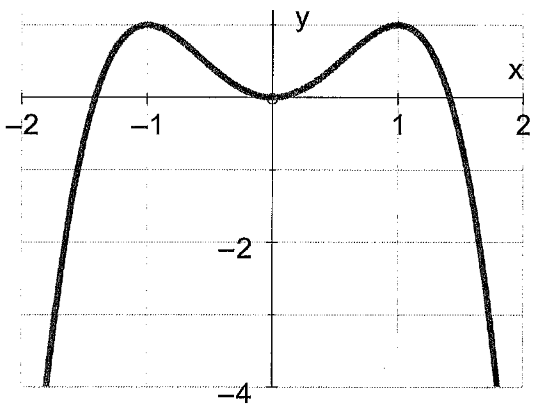
Section 2: Maximum and minimum points

Exercise level 2

1. Find the range of values of x for which $f(x) = x^3 + x^2 - x + 3$ is an increasing function.

2. Find the range of values of x for which $f(x) = x^3 - 6x^2 + 9x + 5$ is a decreasing function.

3. Copy the curve shown below, and sketch the shape of the derivative on the same axes.



4. The equation of a curve is given by $y = 2x + x^2 - 4x^3$.

   (a) Find the coordinates of the turning points on the curve, and distinguish between them by considering the gradient on either side of the turning points.

   (b) Sketch the curve marking the turning points and points of intersection with the axes clearly.

5. The curve $y = x^3 + px^2 + q$ has a minimum point at (4, -11). Find the coordinates of the maximum point on the curve.

6. The curve $y = x^3 + ax^2 + bx + c$ passes through the point (1, 1).

   (a) Write down and simplify an equation connecting a, b and c.

   The curve also has turning points when $x = -1$ and when $x = 3$.

   (b) Find two further equations connecting a, b and c.

   (c) Solve the three equations simultaneously to obtain values for a, b and c.

Worked Examples

Worked Examples

Worked Examples

Worked Examples

Worked Examples

Edexcel AS Mathematics: Differentiation

Section 2: Maximum and minimum points

Exercise level 2 solutions

1. $\mathrm{f}(x) = x^3 + x^2 - x + 3$

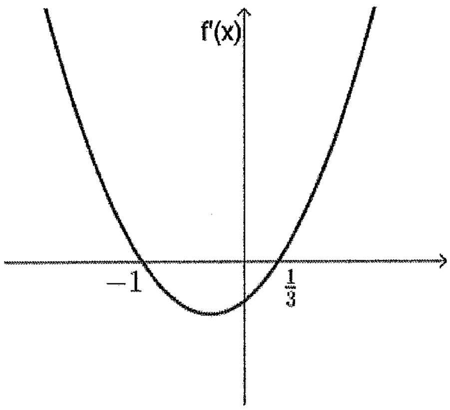
   $\mathrm{f}'(x) = 3x^2 + 2x - 1$

   When f(x) is an increasing function, $\mathrm{f}'(x) > 0$

   $\Rightarrow 3x^2 + 2x - 1 > 0$

   $\Rightarrow (3x - 1)(x + 1) > 0$

   $\Rightarrow x > \frac{1}{3}$ or $x < -1$



   So f(x) is increasing for $x < -1$ and $x > \frac{1}{3}$.

2. $\mathrm{f}(x) = x^3 - 6x^2 + 9x + 5$

   $\mathrm{f}'(x) = 3x^2 - 12x + 9$

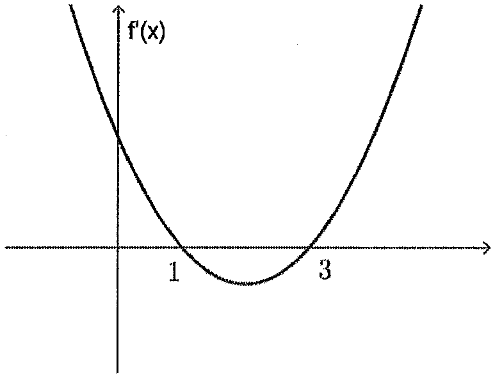
   When f(x) is a decreasing function, $\mathrm{f}'(x) < 0$

   $\Rightarrow 3x^2 - 12x + 9 < 0$

   $\Rightarrow x^2 - 4x + 3 < 0$

   $\Rightarrow (x - 1)(x - 3) > 0$

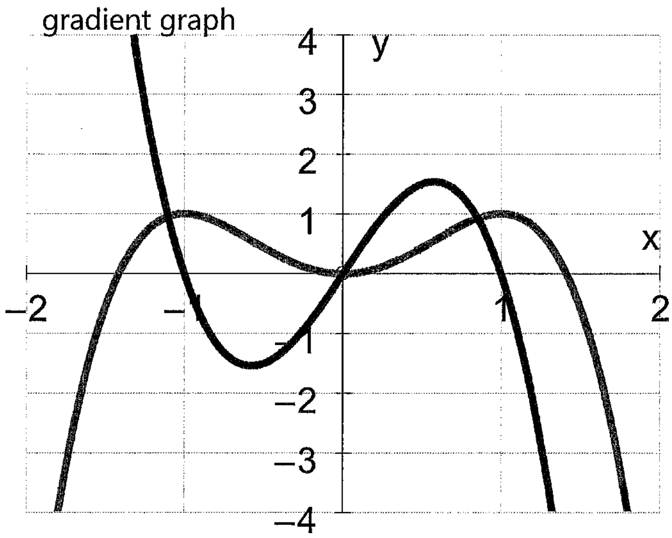
   $\Rightarrow 1 < x < 3$



So f(x) is decreasing for $1 < x < 3$.

3.

The gradient begins as positive, and the graph becomes less steep, so the gradient decreases, reaching zero at $x = -1$ where there is a local maximum point. The gradient then becomes negative, first becoming steeper (more negative) and then less steep (less negative) until the gradient is zero again at the origin where there is a local minimum point. The gradient is then positive, becoming steeper (more positive) and then less steep (less positive) until it again becomes zero at $x = 1$ where there is another local maximum point. After this the gradient becomes negative again, getting steeper so more negative.



4. (a) $y = 2x + x^2 - 4x^3$

$\dfrac{dy}{dx} = 2 + 2x - 12x^2$

At turning points, $\dfrac{dy}{dx} = 0$

$$2 + 2x - 12x^2 = 0$$
$$1 + x - 6x^2 = 0$$
$$6x^2 - x - 1 = 0$$
$$(3x+1)(2x-1) = 0$$
$$x = -\tfrac{1}{3} \text{ or } x = \tfrac{1}{2}$$

When $x = -\frac{1}{3}$, $y = 2\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^2 - 4\left(-\frac{1}{3}\right)^3 = -\frac{2}{3} + \frac{1}{9} + \frac{4}{27} = \frac{-18+3+4}{27} = -\frac{11}{27}$

When $x = \frac{1}{2}$, $y = 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)^3 = 1 + \frac{1}{4} - \frac{1}{2} = \frac{3}{4}$

The turning points are $\left(-\dfrac{1}{3}, -\dfrac{11}{27}\right)$ and $\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$.

x	$x < -\frac{1}{3}$	$x = -\frac{1}{3}$	$-\frac{1}{3} < x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$\dfrac{dy}{dx}$	-ve ↘	0 —	+ve ↗	0 —	-ve ↘

$\left(-\dfrac{1}{3}, -\dfrac{11}{27}\right)$ is a minimum point.

$\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$ is a maximum point.

(b) $y = 2x + x^2 - 4x^3$

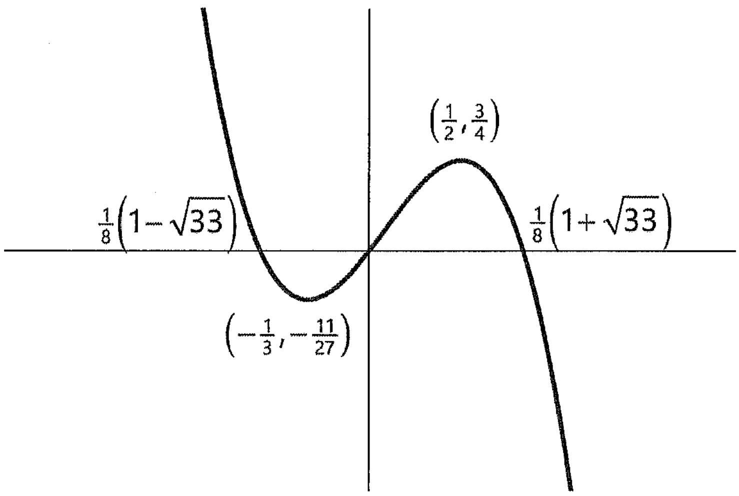
$= x(2 + x - 4x^2)$

$= -x(4x^2 - x - 2)$

The curve cuts the x-axis at x = 0 and at the points satisfying $4x^2 - x - 2 = 0$.

For this quadratic equation, $a = 4, b = -1, c = -2$

Using the quadratic formula, $x = \dfrac{1 \pm \sqrt{1 - 4 \times 4 \times -2}}{8} = \dfrac{1 \pm \sqrt{33}}{8}$



5. $y = x^3 + px^2 + q$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2px$

At turning points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$3x^2 + 2px = 0$

$x(3x + 2p) = 0$

$x = 0 \text{ or } x = -\frac{2p}{3}$

Since there is a minimum point at x = 4, $-\frac{2p}{3} = 4 \quad \Rightarrow p = -6$

The curve is therefore $y = x^3 - 6x^2 + q$.

The point (4, -11) lies on the curve, so $-11 = 4^3 - 6 \times 4^2 + q$

$-11 = 64 - 96 + q$

$q = 21$

The equation of the curve is $y = x^3 - 6x^2 + 21$.

The other turning point is at x = 0, so the maximum point is (0, 21).

6. (a) $y = x^3 + ax^2 + bx + c$

The graph passes through the point (1, 1)

so $1 = 1 + a + b + c$

$a + b + c = 0$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2ax + b$

Turning points are when $3x^2 + 2ax + b = 0$

There is a turning point when x = -1, so $3(-1)^2 + 2a \times -1 + b = 0$

$3 - 2a + b = 0$

$2a - b = 3$

There is a turning point when x = 3, so $3 \times 3^2 + 2a \times 3 + b = 0$

$27 + 6a + b = 0$

$6a + b = -27$

(c) $a + b + c = 0$ (1)

$2a - b = 3$ (2)

$6a + b = -27$ (3)

Adding (2) and (3): $8a = -24 \quad \Rightarrow a = -3$

Substituting into (2) gives: $b = 2a - 3 = -6 - 3 = -9$

Substituting into (1) gives: $c = -a - b = 9 + 3 = 12$

$a = -3, b = -9, c = 12$