Name ....

Further Maths

Integration

Summer Workbook

Developed by MEI Mathematics Education Innovation®

integral

Edexcel AS Mathematics: Integration

Section 1: Introduction to integration

Notes and Examples

These notes contain the following subsections:

Reversing differentiation

Integration is the reverse of differentiation. If you are given an expression for $\frac{dy}{dx}$, and you want to find an expression for y, you need to use integration. This is sometimes called solving a differential equation.

If you differentiate $y = x^2$, you get $\frac{dy}{dx} = 2x$. So you can reverse this process, and integrate $\frac{dy}{dx} = 2x$ to give $y = x^2$. However, notice that differentiating $y = x^2 + 1$ would also give $\frac{dy}{dx} = 2x$, and so would differentiating any expression of the form $y = x^2 + c$. So integrating $\frac{dy}{dx} = 2x$ gives $y = x^2 + c$, where c is called an arbitrary constant. Remember that when you integrate, you must always add an arbitrary constant.

Example 1 shows how you can integrate a function by thinking about what function you would need to differentiate to obtain the given function.

Example 1

Find y as a function of x for each of the following.

(a) $\frac{dy}{dx} = x^3$ (b) $\frac{dy}{dx} = x^6$ (c) $\frac{dy}{dx} = x$

(d) $\frac{dy}{dx} = 2x^2$ (e) $\frac{dy}{dx} = 3x^4$ (f) $\frac{dy}{dx} = 5$

Solution

(a) $\frac{dy}{dx} = x^3 \Rightarrow y = \frac{1}{4}x^4 + c$

> The derivative of x^4 is $4x^3$.
> So integrating $4x^3$ gives x^4.

(b) $\frac{dy}{dx} = x^6 \Rightarrow y = \frac{1}{7}x^7 + c$

> The derivative of x^7 is $7x^6$.
> So integrating $7x^6$ gives x^7.

(c) $\frac{dy}{dx} = x \Rightarrow y = \frac{1}{2}x^2 + c$

> The derivative of x^2 is $2x$.
> So integrating $2x$ gives x^2. Therefore integrating x gives $\frac{1}{2}x^2$.

(d) $\frac{dy}{dx} = 2x^2 \Rightarrow y = \frac{2}{3}x^3 + c$

> The derivative of x^3 is $3x^2$.
> So integrating $3x^2$ gives x^3.
> Therefore integrating x^2 gives $\frac{1}{3}x^3$, and integrating $2x^2$ gives $\frac{2}{3}x^3$.

(e) $\frac{dy}{dx} = 3x^4 \Rightarrow y = \frac{3}{5}x^5 + c$

> The derivative of x^5 is $5x^4$.
> So integrating $5x^4$ gives x^5.
> Therefore integrating x^4 gives $\frac{1}{5}x^5$, and integrating $3x^4$ gives $\frac{3}{5}x^5$.

(f) $\frac{dy}{dx} = 5 \Rightarrow y = 5x + c$

> The derivative of x is 1.
> So integrating 1 gives x.
> Therefore integrating 5 gives $5x$.

The rule for integrating polynomials

The rules for integrating any polynomial function can be summed up as:

Integrating x^n, where n is a positive integer, gives $\frac{x^{n+1}}{n+1}$

Integrating kx^n, where n is a positive integer and k is a constant, gives $\frac{kx^{n+1}}{n+1}$

You can integrate the sum of any number of such functions by simply integrating one term at a time.

Indefinite integration: formal notation

In Example 1 an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ was given and used to find an expression for y.

So you would write:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \Rightarrow y = x^2 + c.$$

Using formal notation, you would write this as:

$$\int 2x \, \mathrm{d}x = x^2 + c$$

You would read this as "the integral of $2x$ with respect to x"

The next example shows integration expressed using formal notation.

Example 2

Integrate each of the following functions.

(a) $x^3 + 3x + 2$

(b) $4x^2 - 5x - 1$

(c) $(x+3)(x-2)$

Solution

(a) $\int \left(x^3 + 3x + 2\right)\mathrm{d}x = \frac{1}{4}x^4 + \frac{3}{2}x^2 + 2x + c$

Remember the arbitrary constant

(b) $\int\left(4x^2-5x-1\right)\mathrm{d}x=\frac{4}{3}x^3-\frac{5}{2}x^2-x+c$

(c) $\int(x+3)(x-2)\ \mathrm{d}x=\int\left(x^2+x-6\right)\ \mathrm{d}x=\frac{1}{3}x^3+\frac{1}{2}x^2-6x+c$

Just as with differentiating, you need to expand the brackets before integrating

Finding the arbitrary constant

If you are given additional information, you can find the value of the arbitrary constant by substituting the given information. This is sometimes called finding the particular solution of a differential equation. The next example shows how this is done.

Example 3

The gradient of a curve at any point (x, y) is given by $\frac{\mathrm{d}y}{\mathrm{d}x}=x^2(2x+1)$.

The curve passes through the point (1, 5).

Find the equation of the curve.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x}=x^2(2x+1)=2x^3+2x$$

Integrating: $y=2\times\frac{1}{4}x^4+\frac{1}{3}x^3+c$

$=\frac{1}{2}x^4+\frac{1}{3}x^3+c$

Substitute the given values of x and y

When $x=1, y=5 \Rightarrow 5=\frac{1}{2}+\frac{1}{3}+c$

$\Rightarrow c=5-\frac{1}{2}-\frac{1}{3}=\frac{25}{6}$

So the equation of the curve is $y=\frac{1}{2}x^4+\frac{1}{3}x^3+\frac{25}{6}$

Edexcel AS Mathematics: Integration

Section 1: Introduction to integration

Exercise level 1

1. Find the following indefinite integrals.

   (a) $\int(2x+3)\,dx$

   (b) $\int(x^2-4x-1)\,dx$

   (c) $\int(x^5+1)\,dx$

   (d) $\int(x^3+2x-7)\,dx$

   (e) $\int(3x-1)^2\,dx$

   (f) $\int x(3x-4)\,dx$

2. A curve has gradient function $\frac{dy}{dx}=3x^2-4$.

   (a) Find an expression for y in terms of x.

   (b) Given that the curve passes through the point (2, -1), find the equation of the curve.

   (c) Show that this curve also passes through the point (1, -4).

3. The gradient function of a curve is given by $\frac{dy}{dx}=4x-x^2$. Find the equation of the curve given that it passes through the point (3, 2).

4. A stone is thrown vertically upwards such that $\frac{dh}{dt}=25-10t$, where t is the time in seconds and h is the height of the stone in metres.
   Given that when $t=0$, $h=30$, find the value of t for which $h=0$.

5. Find y in terms of x given that $\frac{dy}{dx}=(x+1)^2$ and that $y=0$ when $x=2$.

Worked Examples

Worked Examples

Worked Examples

Edexcel AS Mathematics: Integration

Section 1: Introduction to integration

Exercise level 1 solutions

1. (a) $\int(2x+3)\,dx = x^2+3x+c$

(b) $\int(x^2-4x-1)\,dx = \frac{1}{3}x^3-2x^2-x+c$

(c) $\int(x^5+1)\,dx = \frac{1}{6}x^6+x+c$

(d) $\int(x^3+2x-7)\,dx = \frac{1}{4}x^4+x^2-7x+c$

(e) $\int(3x-1)^2\,dx = \int(9x^2-6x+1)\,dx$
$= 3x^3-3x^2+x+c$

(f) $\int x(3x-4)\,dx = \int(3x^2-4x)\,dx$
$= x^3-2x^2+c$

2. (a) $\frac{dy}{dx} = 3x^2-4$
$y = 3\times\frac{1}{3}x^3-4x+c$
$= x^3-4x+c$

(b) When $x = 2$, $y = -1$
$-1 = 2^3-4\times2+c$
$-1 = 8-8+c$
$c = -1$
Equation of curve is $y = x^3-4x-1$

(c) When $x = 1$, $y = 1^3-4\times1-1 = 1-4-1 = -4$
so the curve passes through the point (1, -4).

3. $\dfrac{dy}{dx} = 4x - x^2$

$y = 4 \times \frac{1}{2}x^2 - \frac{1}{3}x^3 + c$

$= 2x^2 - \frac{1}{3}x^3 + c$

When $x = 3$, $y = 2$

$2 = 2 \times 3^2 - \frac{1}{3} \times 3^3 + c$

$2 = 18 - 9 + c$

$c = 2 - 18 + 9 = -7$

$y = 2x^2 - \frac{1}{3}x^3 - 7$

4. $\dfrac{dh}{dt} = 25 - 10t$

$h = 25t - 5t^2 + c$

When $t = 0$, $h = 30 \Rightarrow 30 = c$

$h = 25t - 5t^2 + 30$

When $h = 0$, $25t - 5t^2 + 30 = 0$

$5t - t^2 + 6 = 0$

$t^2 - 5t - 6 = 0$

$(t - 6)(t + 1) = 0$

$t = 6$ or $t = -1$

Since t must be positive, the value of t must be 6.

5. $\dfrac{dy}{dx} = (x+1)^2 = x^2 + 2x + 1$

$y = \frac{1}{3}x^3 + x^2 + x + c$

When $x = 2$, $y = 0$

$0 = \frac{1}{3} \times 2^3 + 2^2 + 2 + c$

$0 = \frac{8}{3} + 4 + 2 + c$

$c = -\frac{26}{3}$

$y = \frac{1}{3}x^3 + x^2 + x - \frac{26}{3}$

Section 1: Introduction to integration

Crucial points

1. **Don't muddle up the formulae for differentiation and integration**

   Example: Given that $\frac{dy}{dx} = x^3$, find y

   I'm being asked to differentiate ✗

✗ ***Wrong*** $\frac{dy}{dx} = x^3 \Rightarrow y = 3x^2$ ✗

The derivative of y is x^3, so to find y I need to integrate ✓

✓ ***Right*** $\frac{dy}{dx} = x^3 \Rightarrow y = \frac{x^4}{4} + c$ ✓

2. **Remember the arbitrary constant *c***
   Always remember to put in the arbitrary constant – you will lose marks in an examination if you miss it out!

Edexcel AS Mathematics: Integration

Section 1: Introduction to integration

Exercise level 2

1. The gradient function of a curve is given by $\frac{dy}{dx} = 4x^2 + x$.

   (a) Find the equation of the curve given that $y = 2$ when $x = 1$.

   (b) Find the value of y when $x = 3$.

2. The gradient of a curve at the point (x, y) is given by $4(1 - x)$. Given that the curve has a maximum value of $y = 8$, find the equation of the curve.

3. Find an expression for y in terms of x if $\frac{dy}{dx} = (x-1)(3x-5)$ and $y = 2$ when $x = 1$.

4. A curve with gradient function $\frac{dy}{dx} = 4x^2 - 1$ has a local minimum value of $y = 1$. Find the equation of the curve and the coordinates of the local maximum point.

5. A curve has gradient function $\frac{dy}{dx} = 3x^2 - 2x + k$.

   (a) The curve has a local maximum point at $x = -2$. Find the value of k.

   (b) The curve passes through the point (1, 3). Find the equation of the curve.

Worked Examples

Worked Examples

Worked Examples

Worked Examples

Edexcel AS Mathematics: Integration

Section 1: Introduction to integration

Exercise level 2 solutions

1. (a) $\frac{dy}{dx} = 4x^2 + x$

$y = \frac{4}{3}x^3 + \frac{1}{2}x^2 + c$

When $x = 1$, $y = 2$

$2 = \frac{4}{3} \times 1^3 + \frac{1}{2} \times 1^2 + c$

$c = 2 - \frac{4}{3} - \frac{1}{2} = \frac{1}{6}$

$y = \frac{4}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}$

(b) When $x = 3$, $y = \frac{4}{3} \times 3^3 + \frac{1}{2} \times 3^2 + \frac{1}{6}$

$= 36 + \frac{9}{2} + \frac{1}{6}$

$= 40\frac{2}{3}$

2. $\frac{dy}{dx} = 4(1 - x)$

At maximum point, $\frac{dy}{dx} = 0 \quad \Rightarrow x = 1$

So the curve passes through the point (1, 8).

$\frac{dy}{dx} = 4(1 - x) = 4 - 4x$

$y = 4x - 2x^2 + c$

When $x = 1$, $y = 8$

$8 = 4 - 2 + c \Rightarrow c = 6$

The equation of the curve is $y = 4x - 2x^2 + 6$.

3. $\frac{dy}{dx} = (x - 1)(3x - 5) = 3x^2 - 8x + 5$

$y = 3 \times \frac{1}{3}x^3 - 8 \times \frac{1}{2}x^2 + 5x + c$

$= x^3 - 4x^2 + 5x + c$

When $x = 1$, $y = 2$

$2 = 1^3 - 4 \times 1^2 + 5 \times 1 + c$

$c = 2 - 1 + 4 - 5 = 0$

$y = x^3 - 4x^2 + 5x$

4. At turning points, $4x^2 - 1 = 0$

$(2x-1)(2x+1) = 0$

$x = \frac{1}{2}$ or $-\frac{1}{2}$

For $x < -\frac{1}{2}$, $\frac{dy}{dx} > 0$

For $-\frac{1}{2} < x < \frac{1}{2}$, $\frac{dy}{dx} < 0$

For $x > \frac{1}{2}$, $\frac{dy}{dx} > 0$

Therefore there is a maximum point where $x = -\frac{1}{2}$ and a minimum point where $x = \frac{1}{2}$.

So the curve passes through the point $\left(\frac{1}{2}, 1\right)$.

$\frac{dy}{dx} = 4x^2 - 1$

$y = \frac{4}{3}x^3 - x + c$

When $x = \frac{1}{2}, y = 1 \Rightarrow 1 = \frac{4}{3}(\frac{1}{2})^3 - \frac{1}{2} + c$

$\Rightarrow 1 = \frac{1}{6} - \frac{1}{2} + c$

$\Rightarrow c = \frac{4}{3}$

The equation of the curve is $y = \frac{4}{3}x^3 - x + \frac{4}{3}$.

The maximum point is when $x = -\frac{1}{2}$.

$y = \frac{4}{3}(-\frac{1}{2})^3 - (-\frac{1}{2}) + \frac{4}{3}$

$= -\frac{1}{6} + \frac{1}{2} + \frac{4}{3}$

$= \frac{5}{3}$

5. (a) For $x = -2$, $\frac{dy}{dx} = 0 \Rightarrow 12 + 4 + k = 0$

$\Rightarrow k = -16$

and so $\frac{dy}{dx} = 3x^2 - 2x - 16$

(b) $\frac{dy}{dx} = 3x^2 - 2x + 8 \Rightarrow y = x^3 - x^2 - 16x + c$

and since the curve passes through (1, 3)

$3 = 1 - 1 - 16 + c \Rightarrow c = 19$

so the equation of the curve is $y = x^3 - x^2 - 16x + 19$